

On competition graphs of n -tuply partial orders

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Abstract

Studying competition graphs of interesting digraphs is a basic open problem in the study of competition graphs. In this context, Cho and Kim [1] studied competition graphs of doubly partial orders and gave a nice characterization of the competition graphs of doubly partial orders. In this paper, we extend their results to a general case, which turns out to be quite interesting. Especially, we take a close look at the competition graphs of triply partial orders to present their meaningful properties.

Keywords: competition graph, doubly partial order; triply partial order; n -tuply partial order; a family of interiors of equilateral triangles; a family of interiors of regular $(n - 1)$ -simplices; planar graphs.

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1 Introduction

Given a digraph $D = (V, A)$, the *competition graph*, denoted by $C(D)$, of D has the same vertex set as D and has an edge xy if for some vertex $u \in V$, the arcs (x, u) and (y, u) are in D . The notion of competition graph is due to Cohen [2] and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modelling of complex economic systems. (See [20] and [22] for a summary of these applications and [6] for a sample paper on the modelling application.) Since Cohen introduced the notion of competition graph, various variations have been defined and studied by many authors (see the survey articles by Kim [10] and Lundgren [15]).

We say that a graph G is an *interval graph* if it is the intersection graph of some family of intervals on the real line. Cohen [2, 3] observed empirically that most competition graphs of

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acyclic digraphs representing food webs are interval graphs. Roberts [21] asked whether or not Cohen's observation was just an artifact of the construction and concluded that it was not by showing that if G is an arbitrary graph, then G together with as many isolated vertices as the edges of G is the competition graph of an acyclic digraph D . (Add a vertex i_α corresponding to each edge $\alpha = \{a, b\}$ of G , and draw arcs from a and b to i_α .) He then asked for a characterization of an acyclic digraph D whose competition graph $C(D)$ is an interval graph. Since then, the problem has remained elusive and it has been the basic open problem in the study of competition graphs. There have been efforts in settling the problem and some progress has been made. Cohen [3] approached the problem from a statistical point of view, trying to build statistical models for the construction of D so that $C(D)$ is (likely to be) an interval graph. Steif [23] showed that there could be no forbidden subgraph characterization of acyclic digraphs whose competition graphs are interval. Lundgren and Maybee [16] gave some results which characterize such a digraph D . But these results essentially boiled down to calculating $C(D)$ and using one of well-known (and efficient) characterizations of an interval graph. While this solves the problem, it is not what we want: a characterization in terms of properties of D . Since the general problem of characterizing acyclic digraphs whose competition graphs are interval seems difficult, Hefner et al. [7] attacked it by putting a constraint on both the indegrees and outdegrees of D .

The study on acyclic digraphs whose competition graphs are interval led to several new problems and applications (see [4,5,13,17] for sample papers). In the same vein, Cho and Kim [1] studied the competition graphs of doubly partial orders and showed that the competition graph of a doubly partial order is an interval graph and any interval graph can be made into the competition graph of a doubly partial order by adding sufficiently many isolated vertices.

A relation $\prec$ on a subset S of $\mathbb{R}^2$ is a *doubly partial order* on S if $(x, y) \prec (z, w)$ whenever $x < z$ and $y < w$ for $(x, y), (z, w) \in S$. A digraph D is called a *doubly partial order* if D is isomorphic to a digraph of a doubly partial order relation $\prec$ on a subset of $\mathbb{R}^2$ (see Figure 1 for an example).

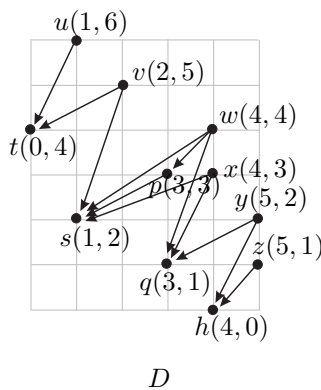


Figure 1: a doubly partial order D and a doubly partial order on a subset of $\mathbb{R}^2$ which is isomorphic to D given in [1]

We say that a graph G is the *competition graph* of a doubly partial order if there is a doubly

partial order D such that G is isomorphic to the competition graph of D .

Theorem 1.1 ([1]). *The competition graph of a doubly partial order is an interval graph.*

Theorem 1.2 ([1]). *Every interval graph can be made into the competition graph of a doubly partial order by adding sufficiently many isolated vertices.*

Various variants of competition graphs of doubly partial orders also have been studied (see [9, 11, 12, 14, 18, 19]).

The concept of doubly partial order can be extended to ‘ n -tuply partial order’ as follows. A relation $\swarrow$ on a subset S of $\mathbb{R}^n$ is an n -tuply partial order on S if, for $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in S$, $(x_1, x_2, \dots, x_n) \swarrow (y_1, y_2, \dots, y_n)$ whenever $x_i < y_i$ for each $i = 1, \dots, n$. For an integer $n \geq 2$, a digraph D is called an n -tuply partial order if D is isomorphic to a digraph of a doubly partial order relation $\swarrow$ on a subset of $\mathbb{R}^n$. We say that a graph G is the *competition graph of an n -tuply partial order* if there is an n -tuply partial order D such that G is isomorphic to the competition graph of D . Thus, if G is the competition graph of an n -tuply partial order, then each vertex of G corresponds to a point in $\mathbb{R}^n$. We sometimes identify a vertex of G with the point corresponding to it without specifically stating it.

In this paper, we present a characterization of the competition graph of an n -tuply partial order, which generalizes the results of Cho and Kim [1], and take a closer look at competition graphs of triply partial orders.

2 Competition graphs of n -tuply partial orders

2.1 $A_{n-1}(\mathbf{p})$ and $\Delta^{(n-1)}(\mathbf{p})$ for a point $\mathbf{p}$ in $\mathbb{R}^n$ the sum of whose components is positive

For simplicity, we use a bold faced letter to represent a point in $\mathbb{R}^n$ ($n \geq 2$) and, for $\mathbf{x} \in \mathbb{R}^n$, we let x_i denote the i th component of $\mathbf{x}$ for each $i = 1, \dots, n$.

We denote by $\mathcal{P}_n$ the set of all points of the hyperplane $x_1 + \dots + x_n = 0$ and by $A_{n-1}(\mathbf{p})$ the set

$$\{\mathbf{w} \mid \mathbf{w} \in \mathcal{P}_n, \mathbf{w} \swarrow \mathbf{p}\}.$$

See Figure 2 for an illustration of $A_2(\mathbf{p})$ for $\mathbf{p} \in \mathbb{R}^3$ with $p_1 + p_2 + p_3 > 0$.

Now we fix a point $\mathbf{p}$ in $\mathbb{R}^n$ with $p_1 + p_2 + \dots + p_n > 0$. Then $A_{n-1}(\mathbf{p})$ is the intersection of the hyperplane $\mathcal{P}_n$ and the interior of the n -cube

$$\{(x_1, \dots, x_n) \mid -\sum_{1 \leq k \leq n, k \neq i} p_k \leq x_i \leq p_i, 1 \leq i \leq n\}.$$

Therefore $A_{n-1}(\mathbf{p})$ is the interior of a regular $(n-1)$ -simplex

$$\Delta^{n-1}(\mathbf{p}) := \{\lambda_1 \mathbf{v}_1 + \dots + \lambda_n \mathbf{v}_n \mid \sum_{k=1}^n \lambda_k = 1, \lambda_i \geq 0, 1 \leq i \leq n\}$$

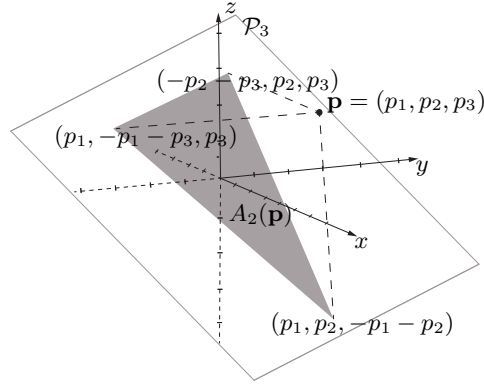


Figure 2: $\mathbf{p} \in \mathbb{R}^3$ and $A_2(\mathbf{p})$.

where $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the intersections of $\mathcal{P}_n$ and the lines going through $\mathbf{p}$ with directional vectors $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$, respectively, and so

$$\begin{aligned} \mathbf{v}_1 &= (p_1, p_2, \dots, -\sum_{1 \leq k \leq n-1} p_k), \\ \mathbf{v}_2 &= (p_1, p_2, \dots, -\sum_{1 \leq k \leq n, k \neq n-1} p_k, p_n), \\ &\vdots \\ \mathbf{v}_n &= (-\sum_{2 \leq k \leq n} p_k, p_2, \dots, p_n). \end{aligned} \tag{1}$$

Note that the length of each edge of $\Delta^{n-1}(\mathbf{p})$ is $\sqrt{2}(p_1 + \dots + p_n)$ and the distance between $\mathbf{p}$ and each vertex of $\Delta^{n-1}(\mathbf{p})$ is $p_1 + \dots + p_n$. Moreover, the directional vector for the line passing through the vertices $\mathbf{v}_i$ and $\mathbf{v}_j$ is

$$(0, \dots, 0, -1, 0, \dots, 0, 1, 0, \dots, 0)$$

if $i < j$ and

$$(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$$

if $i > j$ where -1 and 1 are located in the i th and the j th components for distinct i, j in $\{1, \dots, n\}$. Let S_{n-1} be the set of vectors in $\mathbb{R}^n$ with 0 in the components except the i th and the j th components, -1 in the i th component, and 1 in the j th component for i and j , $1 \leq i < j \leq n$.

The center of $\Delta^{n-1}(\mathbf{p})$ is

$$\begin{aligned} & \frac{1}{n} \left((n-1)p_1 - \sum_{2 \leq k \leq n} p_k, (n-1)p_2 - \sum_{1 \leq k \leq n, k \neq 2} p_k, \dots, (n-1)p_n - \sum_{1 \leq k \leq n-1} p_k \right) \\ &= \left(p_1 - \frac{1}{n} \sum_{k=1}^n p_k, p_2 - \frac{1}{n} \sum_{k=1}^n p_k, \dots, p_n - \frac{1}{n} \sum_{k=1}^n p_k \right) \end{aligned}$$

and the distance between this center and $\mathbf{p}$ is $\frac{1}{\sqrt{n}} \sum_{k=1}^n p_k$ which is $\frac{1}{\sqrt{2n}}$ the edge length of $\Delta^{n-1}(\mathbf{p})$.

We say that two geometric figures in $\mathbb{R}^n$ are *homothetic* if they are related by a geometric contraction or expansion. By the above argument, the following is true.

Proposition 2.1. *Let $\mathbf{p} \in \mathbb{R}^n$ be a point with $\sum_{k=1}^n p_k > 0$. Then $A_{n-1}(\mathbf{p})$ is the interior of a regular $(n-1)$ -simplex $\Delta^{(n-1)}(\mathbf{p})$ with the length of each edge $\sqrt{2} \sum_{k=1}^n p_k$. Furthermore $\mathbf{p}$ is $\frac{1}{\sqrt{2n}}$ the edge length of $\Delta^{n-1}(\mathbf{p})$ away from the center of $\Delta^{(n-1)}(\mathbf{p})$ in the direction $(1, \dots, 1)$. For a point $\mathbf{q}$ in $\mathbb{R}^n$ satisfying $\sum_{k=1}^n q_k > 0$, $A_{n-1}(\mathbf{p})$ and $A_{n-1}(\mathbf{q})$ are homothetic.*

For an illustration for the above proposition, take a point $\mathbf{p}$ in $\mathbb{R}^3$ and project it onto $\mathcal{P}_3$. If (f_1, f_2, f_3) is a foot, then

$$f_1 + f_2 + f_3 = 0$$

and

$$(p_1 - f_1, p_2 - f_2, p_3 - f_3) = t(1, 1, 1)$$

for some real number t . From this, we obtain

$$f_1 = \frac{2p_1 - p_2 - p_3}{3}, f_2 = \frac{2p_2 - p_1 - p_3}{3}, f_3 = \frac{2p_3 - p_1 - p_2}{3}.$$

Thus the foot of $\mathbf{p}$ onto $\mathcal{P}_3$ is the center (of gravity) of $A_2(\mathbf{p})$. We note the distance between (f_1, f_2, f_3) and (p_1, p_2, p_3) is $\frac{p_1 + p_2 + p_3}{\sqrt{3}}$. Therefore, given an equilateral triangle T in $\mathcal{P}_3$ whose sides are on some lines with directional vectors $(-1, 0, 1)$, $(-1, 1, 0)$, $(0, -1, 1)$, the interior of T is represented as $A_2(\mathbf{p})$ for some point $\mathbf{p}$ in $\mathbb{R}^3$ such that the foot of $\mathbf{p}$ onto $\mathcal{P}_3$ is the center of T and $\mathbf{p}$ is away from the center of T at distance $\frac{1}{\sqrt{6}}$ the length of a side.

2.2 A_{n-1} as a bijection from $\{(x_1, x_2, \dots, x_n) \mid x_1 + \dots + x_n > 0\}$ to the set of interiors of certain regular $(n-1)$ -simplices

We start this subsection with the following lemma:

Lemma 2.2. *The vertices of $\Delta^{(n-1)}(1, \dots, 1)$ may be labeled as $\mathbf{w}_1, \dots, \mathbf{w}_n$ so that $\mathbf{w}_j - \mathbf{w}_i$ is a positive scalar multiple of some vector in S_{n-1} if $j > i$ and a negative scalar multiple of some vector in S_{n-1} if $j < i$ and distinct i and j in $\{1, \dots, n\}$. Furthermore,*

$$\mathbf{w}_1 = (1, 1, \dots, -n+1), \mathbf{w}_2 = (1, 1, \dots, -n+1, 1), \dots, \mathbf{w}_n = (-n+1, 1, \dots, 1).$$

Proof. By (1), the coordinates of vertices of $\Delta^{(n-1)}(1, \dots, 1)$ are

$$(1, 1, \dots, -n+1), (1, 1, \dots, -n+1, 1), \dots, (-n+1, 1, \dots, 1).$$

It is easy to check that the labeling

$$\mathbf{w}_1 = (1, 1, \dots, -n+1), \mathbf{w}_2 = (1, 1, \dots, -n+1, 1), \dots, \mathbf{w}_n = (-n+1, 1, \dots, 1)$$

satisfies the condition given in the statement and is the only labeling satisfying the condition. $\square$

Lemma 2.3. *Given points $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbb{R}^n$, suppose that*

$$\Lambda = \left\{ \lambda_1 \mathbf{v}_1 + \dots + \lambda_n \mathbf{v}_n \mid \sum_{k=1}^n \lambda_k = 1, \lambda_i \geq 0, 1 \leq i \leq n \right\}$$

is a regular $(n-1)$ -simplex in the hyperplane $\mathcal{P}_n$ homothetic to $\Delta^{(n-1)}(1, \dots, 1)$. Then, $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbb{R}^n$ may be rearranged so that, for some real numbers $p_1, \dots, p_n$,

$$\begin{aligned} \mathbf{v}_1 &= (p_1, p_2, \dots, -\sum_{1 \leq k \leq n-1} p_k), \\ \mathbf{v}_2 &= (p_1, p_2, \dots, -\sum_{1 \leq k \leq n, k \neq n-1} p_k, p_n), \\ &\vdots \\ \mathbf{v}_n &= (-\sum_{2 \leq k \leq n} p_k, p_2, \dots, p_n). \end{aligned}$$

Proof. Since Λ is homothetic to $\Delta^{(n-1)}(1, \dots, 1)$ satisfying the condition given in Lemma 2.2, the vertices of Λ may be labeled as $\mathbf{u}_1, \dots, \mathbf{u}_n$ so that $\mathbf{w}_j - \mathbf{w}_i$ is a positive scalar multiple of some vector in S_{n-1} if $j > i$ and a negative scalar multiple of some vector in S_{n-1} if $j < i$ and distinct i and j in $\{1, \dots, n\}$. Without loss of generality, we may assume that $\mathbf{v}_1, \dots, \mathbf{v}_n$ is such a labeling.

Since each of the edges of Λ is a part of a line with a directional vector $\mathbf{d}_{ij}$ or $-\mathbf{d}_{ij}$ for some $\mathbf{d}_{ij} \in S_{n-1}$, $\mathbf{v}_i$ and $\mathbf{v}_j$ differ in exactly two components if $i \neq j$. Let $\mathbf{v}_1 = (a_1, \dots, a_n)$. For each integer i , $2 \leq i \leq n$, since $\mathbf{v}_i - \mathbf{v}_1$ is a positive scalar multiple of $\mathbf{d}_i \in S_{n-1}$, $\mathbf{v}_i - \mathbf{v}_1$ is $k_i \mathbf{d}_i$ for some $\mathbf{d}_i \in S_{n-1}$ and a positive real number k_i . Since Λ is regular, $k_1 = \dots = k_n$. We denote k_1 by k for simplicity.

Now fix j , $1 \leq j \leq n$. Then the j th component of $\mathbf{v}_i$ is a_j or $a_j + k$ or $a_j - k$ for each integer i , $1 \leq i \leq n$. Let α_j be the number of vertices in $\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$ the j th component of each of which is different from a_j . If $\alpha_j = 0$, then there are only $\binom{n-1}{2}$ possibilities of directional vectors which are parallel to the edges of Λ , which is impossible. Thus $\alpha_j \geq 1$ for each $j = 1, \dots, n$.

We consider an $(n-1) \times n$ matrix $M = (m_{ij})$ defined by

$$m_{ij} = \begin{cases} 0 & \text{if the } j\text{th component of } \mathbf{v}_{i+1} \text{ equals } a_j; \\ 1 & \text{otherwise} \end{cases}$$

for each $i = 1, \dots, n-1$ and each $j = 1, \dots, n$. Then the sum of the i th row is 2 and the sum of the j th column is α_j for $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Therefore

$$2(n-1) = \sum_{j=1}^n \alpha_j. \tag{2}$$

Then $\alpha_{j_1} = 0$ or 1 for some $j_1 \in \{1, \dots, n\}$. Since $\alpha_{j_1} \geq 1$, $\alpha_{j_1} = 1$. Let $\mathbf{v}_l$ be the vertex whose j_1 th component is different from a_{j_1} . We switch $\mathbf{v}_l$ and $\mathbf{v}_1$ to repeat the above argument. Then $\alpha_{j_1} = n - 1$ and, by (2),

$$n - 1 = \sum_{1 \leq j \leq n, j \neq j_1} \alpha_j.$$

Since $\alpha_j \geq 1$ for each $j = 1, \dots, n$, $\alpha_j = 1$ for $j \neq j_1$. Switching $\mathbf{v}_1$ and $\mathbf{v}_l$ again still preserves the property that $\alpha_r = n - 1$ for some $r \in \{1, \dots, n\}$ and $\alpha_j = 1$ for each $j \neq r$. Suppose that $2 \leq r \leq n - 1$. Since $\alpha_n \geq 1$, $\mathbf{v}_1$ and $\mathbf{v}_s$ differ in the n th component for some $s \in \{2, \dots, n\}$. Then, since $\mathbf{v}_s - \mathbf{v}_1 = k\mathbf{d}_s$ for some $\mathbf{d}_s \in S_{n-1}$, $\mathbf{v}_s$ has $a_r - k$ and $a_n + k$ at the r th component and the n th component, respectively. Since $r \geq 2$, the first component of $\mathbf{v}_t$ is $a_1 + k$ for some $t \in \{2, \dots, n\}$, which contradicts the hypothesis that $\mathbf{v}_t - \mathbf{v}_1 = k\mathbf{d}_t$ for some $\mathbf{d}_t \in S_{n-1}$. Thus either $r = 1$ and the first component of $\mathbf{v}_2, \dots, \mathbf{v}_n$ is $a_1 - k$ or $r = n$ and the n th component of $\mathbf{v}_2, \dots, \mathbf{v}_n$ is $a_n + k$. Arguing inductively, we may conclude that

$$\begin{aligned} \mathbf{v}_1 &= (a_1, a_2, \dots, a_n) \\ \mathbf{v}_2 &= (a_1 - k, a_2 + k, \dots, a_{n-1}, a_n), \\ &\vdots \\ \mathbf{v}_{n-1} &= (a_1 - k, a_2, \dots, a_{n-1} + k, a_n) \\ \mathbf{v}_n &= (a_1 - k, a_2, \dots, a_{n-1}, a_n + k). \end{aligned}$$

or

$$\begin{aligned} \mathbf{v}_1 &= (a_1, a_2, \dots, a_n) \\ \mathbf{v}_2 &= (a_1, a_2, \dots, a_{n-1} - k, a_n + k), \\ &\vdots \\ \mathbf{v}_{n-1} &= (a_1, a_2 - k, \dots, a_{n-1}, a_n + k) \\ \mathbf{v}_n &= (a_1 - k, a_2, \dots, a_{n-1}, a_n + k). \end{aligned}$$

In the former, since Λ is homothetic to $\Delta^{(n-1)}(1, \dots, 1)$, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ correspond to $\mathbf{w}_n, \mathbf{w}_{n-1}, \dots, \mathbf{w}_1$ in order. However, $\mathbf{v}_1 - \mathbf{v}_2 = (k, -k, 0, \dots, 0)$ while $\mathbf{w}_n - \mathbf{w}_{n-1} = (-n, n, 0, \dots, 0)$, which is a contradiction to the hypothesis that Λ is homothetic to $\Delta^{(n-1)}(1, \dots, 1)$. In the latter, we let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ correspond to $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ in order, $p_i = a_i$ for each $1 \leq i \leq n - 1$ and $p_n = a_n + k$ to complete the proof. $\square$

Now we take a regular $(n - 1)$ -simplex Λ in the hyperplane $\mathcal{P}_n$ which is homothetic to $\Delta^{(n-1)}(1, \dots, 1)$. Since Λ and $\Delta^{(n-1)}(1, \dots, 1)$ are homothetic, $cI_n(\mathbf{w}_j - \mathbf{w}_i) = \mathbf{v}_j - \mathbf{v}_i$ for each i and each j in $\{1, \dots, n\}$ for some positive real number c where I_n is the identity matrix, the labelings of the vertices of $\Delta^{(n-1)}(1, \dots, 1)$ and Λ are the same as the one in Lemma 2.2 and the one in Lemma 2.3, respectively. Substituting $i = 1$ and $j = 2$ gives $cn = \sum_{k=1}^n p_k$. Thus $\sum_{k=1}^n p_k > 0$ and so

$$\Lambda = \Delta^{(n-1)}(p_1, p_2, \dots, p_n).$$

By Proposition 2.1, we know that for a point $\mathbf{p}$ in $\mathbb{R}^n$ with $\sum_{i=1}^n p_i > 0$, $A_{n-1}(\mathbf{p})$ is the interior of a regular $(n-1)$ -simplex homothetic to $\Delta^{(n-1)}(1, \dots, 1)$. As a matter of fact, by this result, we may regard A_{n-1} as a function from $\{(x_1, x_2, \dots, x_n) \mid x_1 + \dots + x_n > 0\}$ to the set of interiors of regular $(n-1)$ -simplices homothetic to $\Delta^{(n-1)}(1, \dots, 1)$. In this respect, the argument in this section is to show that A_{n-1} has its inverse.

Therefore we have the following proposition.

Proposition 2.4. *For each integer $n \geq 2$, we may regard A_{n-1} as a bijection from the set $\{(x_1, x_2, \dots, x_n) \mid x_1 + \dots + x_n > 0\}$ of points in $\mathbb{R}^n$ above the hyperplane $\mathcal{P}_n$ to the set of interiors of the regular $(n-1)$ -simplices in the hyperplane $\mathcal{P}_n$ which are homothetic to $\Delta^{(n-1)}(1, \dots, 1)$.*

2.3 A characterization of competition graphs of n -tuply partial orders

We make the following useful observation. We say that $A_{n-1}(\mathbf{p})$ is *purely* included in $A_{n-1}(\mathbf{q})$ if $\Delta^{(n-1)}(\mathbf{p})$ is included in $A_{n-1}(\mathbf{q})$.

Proposition 2.5. *Suppose that $\sum_{k=1}^n p_k > 0$ and $\sum_{k=1}^n q_k > 0$ for $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$. Then $A_{n-1}(\mathbf{p})$ is purely included in $A_{n-1}(\mathbf{q})$ if and only if $\mathbf{p} \prec \mathbf{q}$.*

Proof. Suppose $\mathbf{p} \prec \mathbf{q}$ and take a point $\mathbf{a}$ in $\Delta^{(n-1)}(\mathbf{p})$. Then $a_k \leq p_k$ for each $k = 1, \dots, n$. By the assumption that $\mathbf{p} \prec \mathbf{q}$, $a_k < q_k$ for each $k = 1, \dots, n$, that is, $\mathbf{a} \prec \mathbf{q}$. Thus $\Delta^{(n-1)}(\mathbf{p})$ is included in $A_{n-1}(\mathbf{q})$ and hence $A_{n-1}(\mathbf{p})$ is purely included in $A_{n-1}(\mathbf{q})$.

Suppose that $A_{n-1}(\mathbf{p})$ is purely included in $A_{n-1}(\mathbf{q})$. Then, by (1), $(p_1, \dots, -\sum_{k \neq n} p_k)$, $(p_1, \dots, -\sum_{k \neq n-1} p_k, p_n)$, $\dots$, $(-\sum_{k \neq 1} p_k, \dots, p_n)$ are points in $A_{n-1}(\mathbf{q})$. By the definition of $A_{n-1}(\mathbf{q})$,

$$\begin{aligned} (p_1, \dots, -\sum_{k \neq n} p_k) &\prec (q_1, \dots, q_n); \\ (p_1, \dots, -\sum_{k \neq n-1} p_k, p_n) &\prec (q_1, \dots, q_n); \\ &\vdots \\ (-\sum_{k \neq 1} p_k, \dots, p_n) &\prec (q_1, \dots, q_n), \end{aligned}$$

from which $\mathbf{p} \prec \mathbf{q}$ immediately follows. $\square$

In the rest of this paper, ‘include’ means ‘purely include’ unless otherwise stated.

The following result extends Theorems 1.1 and 1.2:

Theorem 2.6. *A graph G is the competition graph of an n -tuply partial order for an integer $n \geq 2$ if and only if there exists a family $\mathcal{F}$ of interiors of regular $(n-1)$ -simplices in $\mathbb{R}^n$ satisfying the following properties:*

- (i) each element of $\mathcal{F}$ is contained in the hyperplane $\mathcal{P}_n$ and homothetic to $A_{(n-1)}(1, \dots, 1)$;
- (ii) there is a one-to-one correspondence between the vertex set of G and $\mathcal{F}$;
- (iii) vertices v and w are adjacent in G if and only if two elements in $\mathcal{F}$ corresponding to v and w have the intersection including an element in $\mathcal{F}$.

Proof. To show the ‘only if’ part, let D be an n -tuply partial order whose competition graph is G . Without loss of generality, we may assume that the sum of components of every vertex of D is positive by translating each of the vertices of D in the same direction and by the same amount since the competition graph is determined only by the adjacency among vertices of D . Consequently $A_{n-1}(\mathbf{v}) \neq \emptyset$ for each vertex $\mathbf{v}$ of D .

We will claim that the family

$$\mathcal{F} = \{A_{n-1}(\mathbf{v}) \mid \mathbf{v} \in V(D)\}$$

satisfies the properties (i), (ii), and (iii). As (i) and (ii) are trivially satisfied, it remains to show that (iii) is satisfied.

Suppose that two vertices $\mathbf{v}$ and $\mathbf{w}$ are adjacent in G . Then there is a vertex $\mathbf{a}$ in D such that $\mathbf{a} \not\prec \mathbf{v}$ and $\mathbf{a} \not\prec \mathbf{w}$. By Proposition 2.5, $A_{n-1}(\mathbf{a}) \subset A_{n-1}(\mathbf{v})$ and $A_{n-1}(\mathbf{a}) \subset A_{n-1}(\mathbf{w})$, so $A_{n-1}(\mathbf{a}) \subset A_{n-1}(\mathbf{v}) \cap A_{n-1}(\mathbf{w})$.

Suppose that both $A_{n-1}(\mathbf{v})$ and $A_{n-1}(\mathbf{w})$ include an element in $\mathcal{F}$. Then it is of the form $A_{n-1}(\mathbf{a})$ for some $\mathbf{a} \in V(D)$. By Proposition 2.5, $\mathbf{a} \not\prec \mathbf{v}$ and $\mathbf{a} \not\prec \mathbf{w}$, so $\mathbf{v}$ and $\mathbf{w}$ are adjacent in $C(D)$, that is, G .

To show the ‘if’ part, suppose that there exists a family $\mathcal{F}$ of interiors of regular $(n-1)$ -simplices satisfying the properties (i), (ii), and (iii). By Proposition 2.4, each element in $\mathcal{F}$ can be represented as $A_{n-1}(\mathbf{p})$ for some $\mathbf{p} \in \mathbb{R}^n$ with $p_1 + \dots + p_n > 0$. Let D be a digraph with vertex set

$$\{\mathbf{p} \mid A_{n-1}(\mathbf{p}) \text{ is the interior of an } (n-1)\text{-simplex in } \mathcal{F}\};$$

with an arc from $\mathbf{p}$ to $\mathbf{q}$ if and only if $A_{n-1}(\mathbf{q}) \subset A_{n-1}(\mathbf{p})$ for distinct vertices $\mathbf{p}$ and $\mathbf{q}$. By Proposition 2.5, $(\mathbf{p}, \mathbf{q}) \in A(D)$ if and only if $\mathbf{q} \not\prec \mathbf{p}$, so D is an n -tuply partial order.

Now take two vertices v and w in G . Then, by the hypothesis and above argument, v and w correspond to some points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$, respectively, so that v and w are adjacent if and only if both $A_{n-1}(\mathbf{p})$ and $A_{n-1}(\mathbf{q})$ include an element in $\mathcal{F}$ that is $A_{n-1}(\mathbf{r})$ for some $\mathbf{r} \in \mathbb{R}^n$. By the way in which D is defined, $(\mathbf{p}, \mathbf{r}) \in A(D)$ and $(\mathbf{q}, \mathbf{r}) \in A(D)$. Consequently, v and w are adjacent if and only if the corresponding vertices $\mathbf{p}$ and $\mathbf{q}$ have a common out-neighbor in D . Hence G is the competition graph of D which has been shown to be an n -tuply partial order. $\square$

If $n = 2$, then the above theorem gives a necessary and sufficient condition for a graph G being the competition graph of a doubly partial order, that is, there exists a family $\mathcal{F}$ of open intervals such that vertices v and w are adjacent in G if and only if the intervals corresponding to v and w have the intersection including an open interval in $\mathcal{F}$. By Theorem 1.1, the existence of a family $\mathcal{F}$ satisfying the condition given in the above theorem implies that G is the intersection graph of a family of intervals.

3 Competition graphs of triply partial orders

In this section, we take a close look at competition graphs of triply partial orders and derive their interesting properties. We call the interior of a triangle an *open* triangle.

As we consider only competition graphs of triply partial orders, we will omit subscript 2 in $A_2(\mathbf{p})$ for each point $\mathbf{p} \in \mathbb{R}^3$ for simplicity throughout this section.

3.1 A relationship between competition graphs of triply partial orders and intersection graphs of families of open homothetic equilateral triangles

The following theorem is a special case of Theorem 2.6 for $n = 3$.

Theorem 3.1 (A special case of Theorem 2.6 for $n = 3$). *A graph G is the competition graph of a triply partial order if and only if there exists a family $\mathcal{F}$ of open equilateral triangles homothetic to $A_2(1, 1, 1)$ in the hyperplane $x + y + z = 0$ such that vertices v and w are adjacent in G if and only if two open triangles in $\mathcal{F}$ corresponding to v and w have the intersection including an open triangle in $\mathcal{F}$.*

We call a family of open equilateral triangles given in the above corollary a *family of open equilateral triangles making G into the competition graph of a triply partial order*. As Theorem 2.6 is true for any hyperplane in $\mathbb{R}^n$, the family of open equilateral triangles in the above statement is not necessarily in the hyperplane $x + y + z = 0$ but in any hyperplane in $\mathbb{R}^3$.

Given a graph G and a vertex v , if we wish to show that G can be made into the competition graph of a triply partial order by adding sufficiently many isolated vertices, we need to let v correspond to a vertex $\mathbf{v} \in \mathbb{R}^3$ satisfying $v_1 + v_2 + v_3 > 0$ to assign to it the open equilateral triangle $A(\mathbf{v})$ in the hyperplane $x + y + z = 0$. For notational convenience, we shall use $A(v)$ for $A(\mathbf{v})$ from now on.

The following proposition extends Theorem 1.2.

Proposition 3.2. *Suppose that a graph G is the intersection graph of a family $\mathcal{S}$ of homothetic open equilateral triangles in a plane. Then G together with sufficiently many isolated vertices is the competition graph of a triply partial order.*

Proof. Whenever triangles T and T' in $\mathcal{S}$ intersect, we add to $\mathcal{S}$ an open equilateral triangle T'' such that T'' is homothetic to T and is included in both T and T' . Since there are only finitely many intersections, we may add such triangles so that none of them includes another one. Then the resulting family $\mathcal{F}$ is obviously a family of open equilateral triangles making G together with sufficiently many isolated vertices into the competition graph of a triply partial order. $\square$

At this point, we confront an interesting question. As mentioned earlier, for $n = 2$, the fact that there exists a family $\mathcal{F}$ of open intervals such that vertices v and w are adjacent in a graph G if and only if the intervals corresponding to v and w have the intersection including an open interval in $\mathcal{F}$ guarantees that G is the intersection graph of a family of intervals. Is this phenomenon going to be still true for $n = 3$? That is, does the fact that there exists a family $\mathcal{F}$ of interiors of equilateral triangles such that vertices v and w are adjacent in a graph

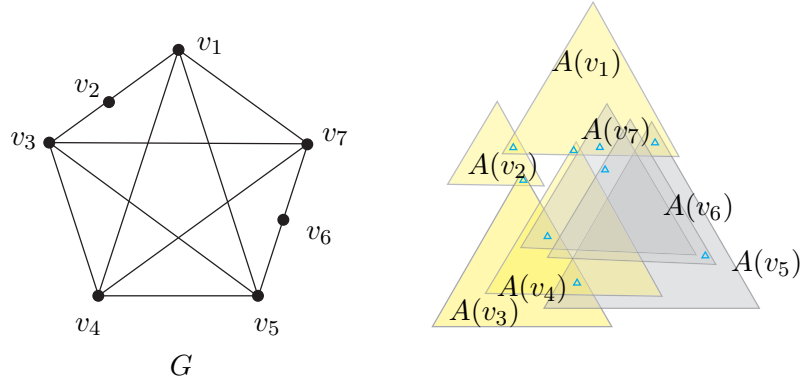


Figure 3: A subdivision G of K_5 and a family of open equilateral triangles making it into the competition graph of a triply partial order

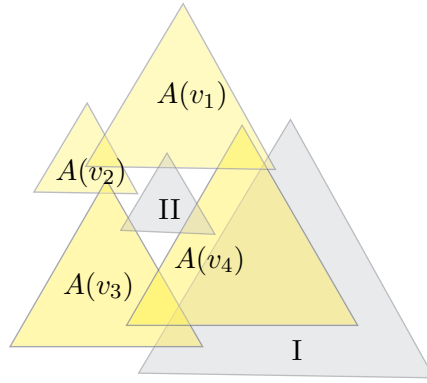


Figure 4: An assignment of equilateral triangles to vertices $v_1, v_2, v_3, v_4, v_5, v_7$ of G given in Figure 3

G if and only if open equilateral triangles in $\mathcal{F}$ corresponding to v and w have the intersection including an element in $\mathcal{F}$ guarantees that G is the intersection graph of a family of homothetic open equilateral triangles?

We will show that there is an example which shows that the answer is no. We consider a subdivision G of K_5 given in Figure 3. There is a family of open equilateral triangles making it into the competition graph of a triply partial order as shown in the figure.

However, G is not the intersection graph of any family of homothetic open equilateral triangles. To see why, we suppose that such a family exists to reach a contradiction. Since $v_1v_2v_3v_4v_1$ is a chordless cycle, $A(v_1)$, $A(v_2)$, $A(v_3)$, $A(v_4)$ are uniquely located as in Figure 4 though the sizes of triangles may vary. We note that v_1, v_3, v_4 are neighbors of both v_5 and v_7 whereas v_2 is not and that v_5 and v_7 are not adjacent. From this observation, we may conclude that the locations of $A(v_5)$ and $A(v_7)$ should be those for the triangles I and II given in Figure 4. Since

the triangle II cannot overlap with $A(v_2)$ and the triangle I, all of its sides are surrounded by $A(v_1)$, $A(v_2)$, $A(v_3)$, and $A(v_4)$. Now, since v_6 is adjacent to v_5 and v_7 , $A(v_6)$ must overlap with both $A(v_5)$ and $A(v_7)$. However, it cannot be done without overlapping one of $A(v_1)$, $A(v_2)$, $A(v_3)$, and $A(v_4)$, which is impossible as none of v_1, v_2, v_3, v_4 is adjacent to v_6 .

3.2 Some families of graphs which can be made into competition graphs of triply partial orders by adding isolated vertices

It is easy to check that the cycle C_n for each $n \geq 3$ can be made into competition graphs of triply partial orders by adding isolated vertices. In the following, we will show that interval graphs and trees can also be made into competition graphs of triply partial orders by adding isolated vertices.

Given homothetic open equilateral triangles $A(v)$, $A(w)$ in a plane, if $A(v)$ includes $A(w)$ or vice versa, then we say that v and w are *inclusion-related*.

Proposition 3.3. *Every interval graph can be made into the competition graph of a triply partial order by adding sufficiently many isolated vertices.*

Proof. Let G be an interval graph. Then there is an interval assignment I_u for each vertex u in G so that $I_v \cap I_w \neq \emptyset$ if and only if $vw \in E(G)$. We draw an open equilateral triangle with I_v as a side for each vertex v in G so that the resulting triangles are homothetic. Whenever two distinct triangles overlap, we draw an open equilateral triangle to be included in the intersection so that the resulting triangles and the ones drawn previously for the vertices of G are homothetic and, for each pair of the resulting triangles, their centers are not inclusion-related.

The family of the equilateral triangles drawn as above is the one making G together with the isolated vertices as many as the triangles which were drawn to be included in the intersection of two triangles into the competition graph of a triply partial order. $\square$

We can also show that any tree can be made into the competition graph of a triply partial order by adding sufficiently many isolated vertices.

Theorem 3.4. *Every tree can be made into the competition graph of a triply partial order by adding sufficiently many isolated vertices.*

Proof. For a technical reason, we assume that any family of homothetic open equilateral triangles making a tree with additional isolated vertices into the competition graphs of a triply partial order is embedded in $\mathbb{R}^2$ and the base of each of them is parallel to the x -axis, which does not lose generality. We call a vertex of each of such triangles the *apex* of the triangle if it is opposite to the base.

We show the following stronger statement by induction on the number of vertices:

For a tree T and a vertex v of T taken as a root, there is a family $\mathcal{F}_v$ of open equilateral triangles making T together with additional isolated vertices into the competition graph of a triply partial order such that, for any vertex w distinct from v , then the apex and the base of $A(w)$ are below the apex and the base of $A(v)$, respectively.

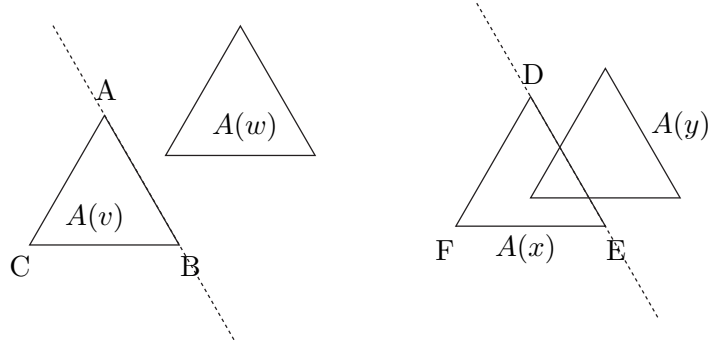


Figure 5: The side AB of $A(v)$ and the side DE of $A(x)$ are separating sides of $\{v, w\}$ and $\{x, y\}$, respectively.

For convenience sake, for a tree T and a vertex v of T taken as a root, we call a family such as $\mathcal{F}_v$ in the above statement a *good family* for T .

If T is a trivial graph, then the statement is vacuously true. Suppose that T has n vertices and any tree on $n - 1$ vertices satisfies the statement for any $n \geq 2$. We fix a vertex v of T as a root. Let $C_1, \dots, C_k$ ($k \geq 1$) be the components of $T - v$. Then $C_1, \dots, C_k$ are trees. Take i from $\{1, \dots, k\}$. We note that C_i has exactly one vertex, say w_i , which was a neighbor of v . We take w_i as a root of C_i .

Now we may apply the induction hypothesis to C_i to obtain a good family $\mathcal{F}_{w_i}$ for C_i for each $i = 1, \dots, k$. Preserving the intersection or the non-intersection of two triangles in $\mathcal{F}_{w_i}$ for each $i = 1, \dots, k$, we may translate the triangles in $\mathcal{F}_{w_1}, \dots, \mathcal{F}_{w_k}$ so that the apexes of $A(w_1), \dots, A(w_k)$ are on the x -axis and any two triangles from distinct families do not intersect. Let δ_i be distance between the apex of $A(w_i)$ and the apex of a triangle which is the second highest among the apexes of the triangles in $\mathcal{F}_{w_i}$. Now we draw $A(v)$ in such a way that the base of $A(v)$ is a part of the line $y = -\delta/2$ and long enough to intersect all of $A(w_1), \dots, A(w_k)$, and add a small triangle $A(a_i)$ to be contained in $A(v) \cap A(w_i)$ for each $i = 1, \dots, k$ where δ is the minimum among $\delta_1, \dots, \delta_k$.

Then $\mathcal{F}_{w_1} \cup \dots \cup \mathcal{F}_{w_k} \cup \{A(v), A(a_1), \dots, A(a_k)\}$ is a good family for T and this completes the proof. $\square$

3.3 Necessary conditions for a graph being the competition graph of a triply partial order

Kaufmann, *et al.* [8] showed that the complement of the cycle C_n cannot be a max-tolerance graph for any $n \geq 10$. By applying an argument motivated by their proof idea, we may show the following theorem.

Theorem 3.5. *The complement of the competition graph of a triply partial order cannot contain the cycle C_n as an induced subgraph for any $n \geq 10$.*

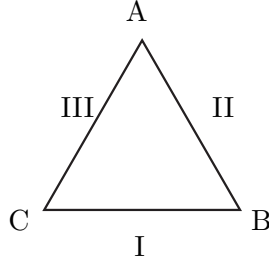


Figure 6: An equilateral triangle ABC

Let G be the competition graph of a triply partial order D . In order to show the above theorem, we need to classify a pair of nonadjacent vertices $\{v, w\}$ of G according to the relative position of $A(v)$ and $A(w)$. Suppose that v and w are not inclusion-related. Then there is a side such that the line containing the side divides the plane into two half planes to have one half plane include one of $A(v)$, $A(w)$, say $A(v)$ and the other half plane include $A(w) \setminus A(v)$ (see Figure 5 for an illustration).

We will call such a side a *separating side* of $\{v, w\}$. It is possible that there may be more than one separating side for a pair of nonadjacent vertices (the pair $\{v_1, v_4\}$ in Figure 7(a) is such an example). Take a separating side for a pair of nonadjacent vertices of G . If it is parallel to the side BC, AB, AC of the equilateral triangle ABC given in Figure 6, then we say that it is of Type I, Type II, Type III, respectively.

Given a family $\mathcal{F}$ of homothetic open equilateral triangles, let us call a triangle in $\mathcal{F}$ which includes no triangle in $\mathcal{F}$ a *minimal triangle* of $\mathcal{F}$.

Lemma 3.6. *Let G be the competition graph of a triply partial order D and $\mathcal{F}$ be a family of open equilateral triangles making G into the competition graph of D . Suppose that $v_1v_2v_3v_4v_1$ is an induced subgraph of G which is isomorphic to C_4 . Then neither v_1 and v_3 nor v_2 and v_4 are inclusion-related. Furthermore, $\{v_1, v_3\}$ and $\{v_2, v_4\}$ do not have separating sides of the same type.*

Proof. Suppose that v_1 and v_3 are inclusion-related. Without loss of generality, we may assume that $A(v_1) \subset A(v_3)$. Since v_1 is adjacent to v_2 , $A(v_1) \cap A(v_2)$ contains a minimal triangle. By the assumption that $A(v_1) \subset A(v_3)$, the minimal triangle is included in $A(v_1) \cap A(v_3)$. Then v_1 and v_3 are adjacent and we reach a contradiction. Thus v_1 and v_3 are not inclusion-related. In a similar way, we can show that v_2 and v_4 are not inclusion-related.

To show the ‘furthermore part’ by contradiction, suppose that $\{v_1, v_3\}$ and $\{v_2, v_4\}$ have separating sides of the same type. We denote by e and f the separating sides of $\{v_1, v_3\}$ and $\{v_2, v_4\}$, respectively. Without loss of generality, we may assume that e and f are of Type I. In addition, we may assume that e and f belong to $A(v_1)$ and $A(v_2)$, respectively. Then e is above the side of $A(v_3)$ parallel to e , and f is above the side of $A(v_4)$ parallel to f . Since $v_1v_2v_3v_4v_1$ is a 4-cycle of G , v_i and v_{i+1} are adjacent and so $A(v_i)$ and $A(v_{i+1})$ overlap for $i = 1, \dots, 4$ where the subscripts are reduced modular 4. If $A(v_1) \cap A(v_3) = \emptyset$, then f is below e since $A(v_2) \cap A(v_3) \neq \emptyset$

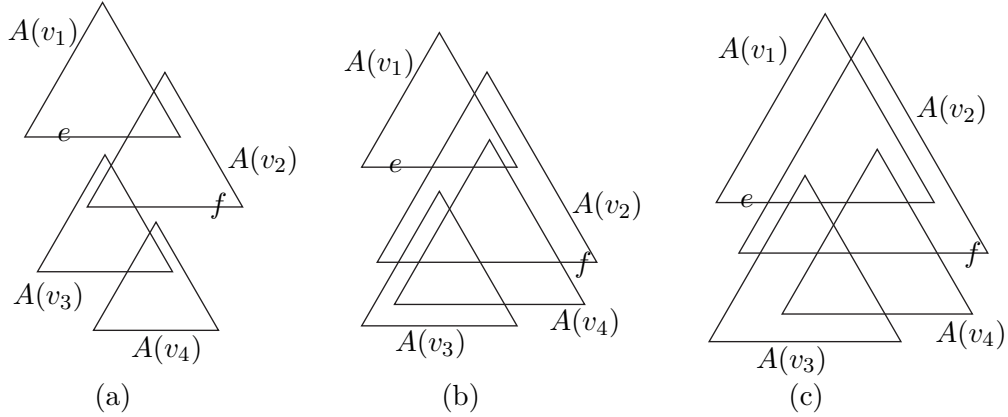


Figure 7: Supplementary figures for the proof of Lemma 3.6

(refer to (a) and (b) of Figure 7). Assume that $A(v_1) \cap A(v_3) \neq \emptyset$ (refer to Figure 7(c)). Suppose, to the contrary, that f is above e . Then $A(v_2) \cap A(v_3)$ is above f . Note that e is above the side of $A(v_3)$ which is parallel to e . Yet, $A(v_1)$ and $A(v_3)$ are not inclusion-related, so $A(v_2) \cap A(v_3)$ is contained in $A(v_1)$. Since v_2 and v_3 are adjacent, $A(v_2) \cap A(v_3)$ includes a minimal triangle T . As we have shown that $A(v_2) \cap A(v_3) \subset A(v_1)$, T is included in $A(v_1)$. Since T is included in $A(v_3)$, v_1 and v_3 are adjacent in G and we reach a contradiction. Hence e is above f in any case.

By applying an argument parallel to the above argument, we can show that f is above e in both cases $A(v_2) \cap A(v_4) = \emptyset$ and $A(v_2) \cap A(v_4) \neq \emptyset$ to reach a contradiction. $\square$

Proof of Theorem 3.5. Let G be the competition graph of a triply partial order D and $\mathcal{F}$ be the set of open equilateral triangles making G into the competition graph of D . Suppose that the complement $\overline{G}$ of G contains an induced subgraph H which is isomorphic to C_n for some positive integer n . Set $H = v_1 v_2 \dots v_n v_1$. Without loss of generality, we may assume that $\{v_1, v_2\}$ has a separating side of Type I. Then v_1 and v_2 are adjacent in $\overline{G}$ and, for each $i \in \{4, \dots, n-2\}$, $v_1 v_i v_2 v_{i+1} v_1$ is an induced 4-cycle of G . Thus, by Lemma 3.6, $\{v_i, v_{i+1}\}$ does not have a separating side of Type I for any $i \in \{4, \dots, n-2\}$. Hence $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$ are the only possible consecutive pairs on the cycle that might have separating sides of Type I. Note that $v_{n-1} v_2 v_n v_3 v_{n-1}$ and $v_n v_3 v_1 v_4 v_n$ are 4-cycles as induced subgraphs of G . Thus, by Lemma 3.6, neither $\{v_2, v_3\}$ and $\{v_{n-1}, v_n\}$ nor $\{v_1, v_n\}$ and $\{v_3, v_4\}$ have separating sides of the same type. Thus at most three pairs have separating sides of Type I. By the symmetric argument, we can show that at most three pairs have separating sides of Type II and at most three pairs have separating sides of Type III. Hence $n \leq 9$. $\square$

By Proposition 3.3, K_5 can be made into the competition graph of a triply partial order by

adding isolated vertices, which implies that the competition graph of a triply partial order may not be planar. However, there is no non-planar K_3 -free graph which is the competition graph of a triply partial order by the following theorem.

Theorem 3.7. *If the competition graph of a triply partial order is K_3 -free, then it is planar.*

Proof. Let G be the competition graph of a triply partial order D which is K_3 -free. By Theorem 3.1, there exists a family $\mathcal{F}$ of open equilateral triangles making G into the competition graph of D . If a triangle in $\mathcal{F}$ intersects with at most one triangle, then we may delete it from $\mathcal{F}$, accordingly deleting an isolated vertex from G without affecting the planarity of G . Therefore we may assume that each triangle in $\mathcal{F}$ intersects with at least two triangles.

If a minimal triangle is included in at most one triangle, then we may delete it without affecting the planarity of G . Thus we may assume that each minimal triangle is included in at least two triangles in $\mathcal{F}$. Yet, since G is K_3 -free, no minimal triangle is included in more than two triangles. Thus

$$\text{a minimal triangle is included in exactly two triangles in } \mathcal{F}. \quad (\dagger)$$

If, for two non-minimal triangles, one is included in the other, then the vertex corresponding to the included one has degree 1 in G since G is K_3 -free. Deleting a vertex of degree 1 from G does not affect the planarity of G , so we may assume that a non-minimal triangle is not included in a triangle.

Take a non-minimal triangle in $\mathcal{F}$. Then it contains at least one minimal triangle. On the other hand, it cannot include a non-minimal triangle by our assumption. Furthermore, if two triangles in $\mathcal{F}$ overlap and a minimal triangle is included in the intersection, then it cannot be included in other triangles by $(\dagger)$. Thus, whenever the intersection of two triangles contains more than one minimal triangle in $\mathcal{F}$, we may delete minimal triangles in the intersection, accordingly deleting isolated vertices from G , to leave exactly one minimal triangle in it without affecting the planarity of G . Furthermore, if a minimal triangle is concurrent with one of the two triangles including it, we may shift it slightly so that it is concurrent with neither of them.

We consider a graph Γ represented in $\mathcal{P}_3$ in the following way. The vertex set of Γ is the set of centers of triangles in $\mathcal{F}$. Let $A(x)$ and $A(y)$ be overlapping non-minimal triangles. Without loss of generality, we may assume that $A(x)$ and $A(y)$ are located as shown in Figure 8(a). Then $A(x) \cap A(y)$ includes a minimal triangle $A(a)$ for some vertex a in G . Now we draw line segments joining x and a and joining y and a , respectively. From now on, we denote by $\overline{vw}$ the line segment joining two vertices v and w for notational convenience. If the center of a minimal triangle T is on one of the line segments, then we may shift T so that T is included in exactly those triangles where T was originally included and the center of T avoids the line segment as shown in (a) and (b) of Figure 8. Suppose the center u of a non-minimal triangle $A(u)$ is on the line segment $\overline{xa}$. Then $A(u)$ cannot include $A(a)$ since G is K_3 -free. Thus the side of $A(u)$ parallel to $\overline{AC}$ is included in $A(x)$. Since u is on $\overline{xa}$, $A(u)$ is included in $A(x)$ and we reach a contradiction. Therefore the center of any non-minimal triangle in $\mathcal{F}$ cannot be on the line segment $\overline{xa}$. By using a similar argument, one can show that the center of any non-minimal triangle in $\mathcal{F}$ cannot be on the line segment $\overline{yb}$ either. Thus, the sequence (x, a, y) corresponds

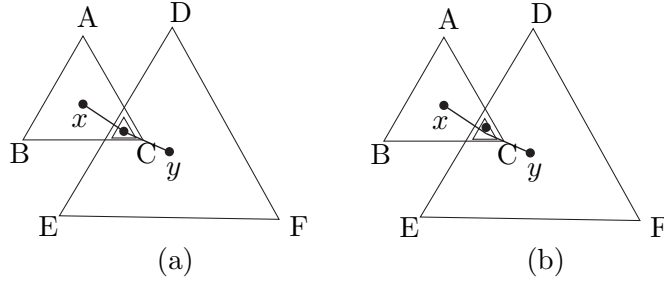


Figure 8: $A(x)$, $A(y)$, and $A(a)$; (a) the path (x, a, y) ; (b) the edge joining x and y .

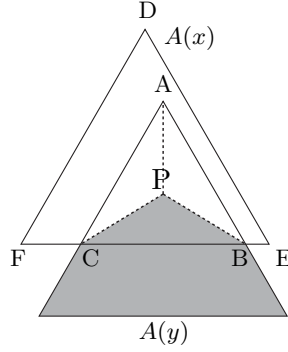


Figure 9: Two overlapping triangles $A(x)$ and $A(y)$

to the path (x, a, y) in Γ . Furthermore, since no minimal triangle is concurrent with a triangle including it, each line segment has a positive length. Now we disregard a to make the path (x, a, y) into one piece of curve segment joining x and y so that it represents the edge joining x and y in Γ . See Figure 8(b) for an illustration. It is easy to check that Γ is isomorphic to G .

To show that Γ is a plane graph, we first show that two line segments joining the center of a non-minimal triangle and the center of a minimal triangle contained in it do not intersect. To do so, we take two non-minimal triangles $A(x)$ and $A(y)$ and two minimal triangles $A(a)$ and $A(b)$ such that $A(a) \subset A(x)$ and $A(b) \subset A(y)$. If $A(a) \subset A(y)$, then $A(a)$ and $A(b)$ coincide by the assumption that at most one minimal triangle exists in the intersection of two non-minimal triangles and so $\overline{xa}$ and $\overline{yb}$ do not intersect. Thus we may assume that $A(a) \not\subset A(y)$. For the same reason, we may assume that $A(b) \not\subset A(x)$. If $A(x)$ and $A(y)$ do not overlap, then $\overline{xa}$ and $\overline{yb}$ clearly do not intersect. Now we suppose that $A(x)$ and $A(y)$ overlap and let A, B, C be the vertices of the equilateral triangle which is the intersection of $A(x)$ and $A(y)$. Without loss of generality, we may assume that $\{x, y\}$ is of Type I (see Figure 9 for an illustration). In addition, let P be the center of the triangle ABC, and D, E, F be the vertices of $A(x)$ as shown in the figure. Suppose that b is in the triangle ACP. Since $A(b) \not\subset A(x)$, the side of $A(b)$ parallel to $\overline{BC}$ is in the exterior of $A(x)$. Since the distance between b and the perpendicular foot of b on $\overline{AC}$ is smaller than the distance between b and the perpendicular foot of b on $\overline{BC}$, the side of

$A(b)$ parallel to $\overline{AC}$ is in the exterior of $A(y)$, which is contradiction. Thus b cannot be in the triangle ACP . By applying a similar argument, we may show that b cannot be in the triangle ABP . Consequently b is in the shaded region in Figure 9. Suppose that a is in the triangle BCP . Since $A(a) \not\subset A(y)$, the side of $A(a)$ parallel to $\overline{AB}$ or the side of $A(a)$ parallel to $\overline{AC}$ is in the exterior of $A(y)$. In either case, the side of $A(a)$ parallel to $\overline{BC}$ is in the exterior of $A(x)$ and we reach a contradiction. Furthermore a cannot be in the exterior of $A(x)$, so a cannot be in the shaded region in Figure 9.

Since y lies on the ray $\overrightarrow{AP}$ and is located below P , y is in the shaded region. Since x lies on the line passing through F and parallel to $\overline{CP}$, x cannot be in the shaded region. Furthermore, since x is the intersection of the two rays which are parallel to $\overline{BP}$ and $\overline{CP}$, respectively, and going through E and F , respectively, $\overline{xa}$ does not intersect the shaded region. Thus we can conclude that $\overline{xa}$ and $\overline{yb}$ do not intersect.

Any edge of Γ is the concatenation of two line segments joining the center of a non-minimal triangle and the center of a minimal triangle, respectively. Hence, by the above argument, any two distinct edges cannot intersect in Γ and so Γ is a plane graph. $\square$

By Kuratowski's theorem, the following corollary is an immediate consequence of Theorem 3.7.

Corollary 3.8. *If a K_3 -free graph has a subdivision of the complete bipartite graph $K_{3,3}$, it cannot be made into the competition graph of a triply partial order even if it is allowed to add isolated vertices.*

Suppose that a graph G has $K_{3,3}$ as an induced subgraph and G is the competition graph of a triply partial order. Then there is a family $\mathcal{F}$ of equilateral triangles making G into the competition graph of a triply partial order. If we take triangles corresponding to the vertices of $K_{3,3}$ and the triangles included in two of those triangles, then these triangles form a family making $K_{3,3}$ together with sufficiently many isolated vertices into the competition graph of a triply partial order, which is impossible by Corollary 3.8. Thus $K_{3,3}$ is a forbidden subgraph for the competition graph of a triply partial order.

Corollary 3.9. *If a graph G is the competition graph of a triply partial order, then G does not contain $K_{3,3}$ as an induced subgraph.*

4 Open questions

We have not found a plane graph which cannot be made into the competition graph of a triply partial order by adding sufficiently many isolated vertices. We conjecture that any plane graph can be made into the competition graph of a triply partial order by adding sufficiently many isolated vertices.

Especially, we would like to see if the converse of Theorem 3.7 is true, that is, if a graph is K_3 -free and planar, then can it be made into the competition graph of a triply partial order by adding isolated vertices?

We suggest an expansion of known family of graphs which can be made into competition graphs of triply partial orders. Chordal graphs seem to be the first graphs to be considered.

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